Uncertainty Analysis of Nonlinear Aeroelastic Systems

Muhammad R. Hajj
ESM Department, Virginia Tech

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Topics

- Stochastic Modeling of Incident Turbulence

  Contributions: C. Pettit, P. Beran, M. Ghommem and I. Puri

- Response of Nonlinear Aeroelastic Systems

  Contributions: A. Nayfeh and M. Ghommem
Stochastic Modeling of Incident Gust Effects on Aerodynamic Lift

- Upstream turbulence or gust are usually specified by many parameters, including turbulence intensities, integral length scales, relative energy content of large and small scales...

- Difficult to perform a parametric assessment of their individual effects on different flow quantities.

  ➢ The need of a stochastic approach to effectively perform sensitivity analysis of flow quantities to variations in gust or turbulence parameters.
The Deterministic Problem

Unsteady Vortex Lattice Model (UVLM).

\[ \mathbf{V}_\infty(t) = [V_\infty + u(t), v(t)]^T \]

\[
L_1 \gamma_a(t) = L_2(r_w(t)) \gamma_w(t) + L_3 \mathbf{V}_\infty(t), \\
\gamma_w(t + 1) = L_4 \gamma_w(t) + L_5(r_w(t)) \gamma_a(t), \text{ and} \\
r_w(t + 1) = L_6(r_w(t)) \gamma_w(t) + L_7(r_w(t)) \gamma_a(t) + L_8 \mathbf{V}_\infty(t),
\]

Sinusoidal $v$ component

Random $u$ and $v$ components
### Uncertain Gust Representation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Mean</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_u$</td>
<td>1.716 ft/s</td>
<td>0.1716 ft/s</td>
</tr>
<tr>
<td>$L_u$</td>
<td>505.2 ft</td>
<td>50.52 ft</td>
</tr>
<tr>
<td>$\sigma_v$</td>
<td>1.000 ft/s</td>
<td>0.1000 ft/s</td>
</tr>
<tr>
<td>$L_v$</td>
<td>50.00 ft</td>
<td>5.000 ft</td>
</tr>
</tbody>
</table>

\[
S_{uu}(\omega) = \frac{2\sigma_u^2 L_u}{\pi V_\infty} \frac{1}{[1 + (1.339 L_u \omega / V_\infty)^2]^{5/6}}
\]

\[
S_{vv}(\omega) = \frac{2\sigma_v^2 L_v}{\pi V_\infty} \frac{1 + \frac{8}{3}(2.678 L_v \omega / V_\infty)^2}{[1 + (2.678 L_v \omega / V_\infty)^2]^{11/6}}
\]

\[
u(t) = \sum_{n=1}^{n=N_\omega} \sqrt{2S_{uu}(\omega_n)\Delta \omega_n(\omega_n)} \cos(\omega_n t + \phi_n), \text{ and}
\]

\[
u(t) = \sum_{n=1}^{n=N_\omega} \sqrt{2S_{vv}(\omega_n)\Delta \omega_n(\omega_n)} \cos(\omega_n t + \phi_n).
\]
Intrusive Formulation

$\gamma_a(t, \xi)$ and $\gamma_w(t, \xi)$ are considered as stochastic processes that are functions of the statistics of the upstream turbulence.

$$
\gamma_a(t, \Xi) = \sum_{m=0}^{P} \gamma_{am}(t)\Psi_m(\Xi), \text{ and}

\gamma_w(t, \Xi) = \sum_{m=0}^{P} \gamma_{wm}(t)\Psi_m(\Xi),
$$

The stochastic version of the UVLM governing equations

$$
L_1\gamma_{al}(t) = L_2(r_w(t))\gamma_{wl}(t) + L_3V_{\infty l}(t), \quad (l = 0, 1, \ldots, P),

\gamma_{wl}(t+1) = L_4 \gamma_{wl}(t) + L_5(r_w(t))\gamma_{al}(t), \quad (l = 0, 1, \ldots, P), \text{ and}

r_w(t+1) = L_6(r_w(t))\gamma_{w0}(t) + L_7(r_w(t))\gamma_{a0}(t) + L_8V_{\infty0}(t).
$$

at $t = 0$ \quad $\gamma_w(0) = 0$ and $r_w(0) = 0$
PCE coefficients of $V_\infty$

Zero- and first-order coefficients
Sensitivity of Lift Coefficient

Zero and first order coefficients

\[ C_L(t, \Xi) = \sum_{m=0}^{P} C_{Lm}(t) \Psi_m(\Xi) \]
Validation

(a) $C_L(t = 1s, \xi)$ vs. $\xi_{\sigma_v}$  
(b) $C_L(t = 1s, \xi)$ vs. $\xi_{L_v}$  
(c) $C_L(t = 1s, \xi)$ vs. $\xi_{\sigma_u}$  
(d) $C_L(t = 1s, \xi)$ vs. $\xi_{L_u}$

Lift coefficient at $t = 1s$ versus $\xi_i$

<table>
<thead>
<tr>
<th></th>
<th>Slope (regression analysis)</th>
<th>The corresponding PCE coefficient (t=1.0s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_L$ vs. $\xi_{\sigma_v}$</td>
<td>-0.033275</td>
<td>$C_{L_1} = -0.0331$</td>
</tr>
<tr>
<td>$C_L$ vs. $\xi_{L_v}$</td>
<td>-0.01261</td>
<td>$C_{L_2} = -0.0125$</td>
</tr>
<tr>
<td>$C_L$ vs. $\xi_{\sigma_u}$</td>
<td>0.00069556</td>
<td>$C_{L_3} = 0.0006911$</td>
</tr>
<tr>
<td>$C_L$ vs. $\xi_{L_u}$</td>
<td>0.001554</td>
<td>$C_{L_4} = 0.0015$</td>
</tr>
</tbody>
</table>
Validation

(a) $t=0.5$ s 

(b) $t=1.0$ s 

(c) $t=1.5$ s 

(d) $t=2.0$ s 

Empirical density functions for $C_L$ at $t=0.5$, 1.0, 1.5 and 2.0 s
Observations

- As expected, lift fluctuations are affected primarily by $\sigma_v$

- Nonintrusive approach was implemented first.
  - Wake vortices

- Second-order PCE yielded errors in estimated lift coefficients

- This work is a first step towards the implementation of a stochastic approach based on PCE to characterize nonlinear fluid-structure interactions problems.
Flutter and LCO of a Pitch-Plunge Airfoil

Competing nonlinearities in aeroelastic systems:
  geometric, inertia, aerodynamics, free-play, etc.

Response:
Multiple equilibria, bifurcations, limit cycles oscillations (LCO), chaos, various types of resonances (internal, super/subharmonic)

➤ Combined effects of these nonlinearities must be considered.

Parameter variations could have substantial impact on the system’s stability and/or response

➤ Effects of parameters variations on the flutter and LCO of an airfoil.
Pitch-Plunge Airfoil – Deterministic Problem

Nonlinear bending and torsional stiffnesses

\[
\begin{align*}
  k_\alpha(\alpha) &= k_{\alpha 0} + k_{\alpha 1} \alpha + k_{\alpha 2} \alpha^2 + ... \\
  k_h(h) &= k_{h 0} + k_{h 1} h + k_{h 2} h^2 + ... 
\end{align*}
\]

Nonlinear quasi-steady aerodynamic loads

\[
\begin{align*}
  L &= \rho U^2 b c_l(\alpha_{eff} - c_s \alpha_{eff}^3) \\
  M &= \rho U^2 b^2 c_m(\alpha_{eff} - c_s \alpha_{eff}^3) \\
  \alpha_{eff} &= (\alpha + \frac{\dot{h}}{U} + \frac{1}{2} - a) b \frac{\dot{\alpha}}{U} \\
  \begin{pmatrix}
    \ddot{h} \\
    \dot{\alpha}
  \end{pmatrix} &= \begin{pmatrix}
    m_T & m_W x_a b \\
    m_W x_a b & I_\alpha
  \end{pmatrix}
  \begin{pmatrix}
    \ddot{h} \\
    \dot{\alpha}
  \end{pmatrix} + \begin{pmatrix}
    c_h & 0 \\
    0 & c_\alpha
  \end{pmatrix}
  \begin{pmatrix}
    \dot{h} \\
    \dot{\alpha}
  \end{pmatrix} + \begin{pmatrix}
    k_h(h) & 0 \\
    0 & k_\alpha(\alpha)
  \end{pmatrix}
  \begin{pmatrix}
    h \\
    \alpha
  \end{pmatrix} = \begin{pmatrix}
    -L \\
    M
  \end{pmatrix}
\end{align*}
\]
Different branches of deterministic LCO solutions (curves)

Supercritical

Subcritical

Stochastic solutions (cross hatched)
Pitch-Plunge Airfoil – Deterministic Problem

The state variables

\[
\begin{align*}
Y &= \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} = \begin{pmatrix} h \\ \alpha \\ \dot{h} \\ \dot{\alpha} \end{pmatrix} \\
\dot{Y} &= F(Y, U)
\end{align*}
\]

where

\[
F(Y, U) = \begin{pmatrix} Y_3 \\ Y_4 \\ -p_h(Y_1)Y_1 - (k_1 U^2 + p_\alpha(Y_2))Y_2 - c_1 Y_3 - c_2 Y_4 + g_{NL1}(Y) \\ -q_h(Y_1)Y_1 - (k_2 U^2 + q_\alpha(Y_2))Y_2 - c_3 Y_3 - c_4 Y_4 + g_{NL2}(Y) \end{pmatrix}
\]

\[
\dot{Y} = \Lambda(U)Y \mid Q(Y, Y) \mid C(Y, Y, Y)
\]
Pitch-Plunge Airfoil – Deterministic Linear Problem

- Linear problem: Flutter Speed

\[
\dot{\mathbf{Y}} = A(U)\mathbf{Y}
\]

where

\[
A(U) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-I_\alpha k_{h0}/d & -(k_1 U^2 - m_W x_\alpha b k_{\alpha_0}/d) & -c_1 & -c_2 \\
m_W x_\alpha b k_{h0}/d & -(k_2 U^2 + m_T k_\alpha_0/d) & -c_3 & -c_4 \\
\end{pmatrix}
\]

Specific airfoil geometry
Pitch-Plunge Airfoil

- Uncertainty in **Flutter Speed** due to variation in torsional stiffness – Intrusive PCE

\[ k_{\alpha 0} = \bar{k}_{\alpha 0} + \sigma_1 \xi_1 \]

\[
\begin{align*}
    h(t, \xi_1) &= \sum_{i=0}^{P} h_i(t) \Psi_i(\xi_1) \\
    \alpha(t, \xi_1) &= \sum_{i=0}^{P} \alpha_i(t) \Psi_i(\xi_1)
\end{align*}
\]

\[
\begin{pmatrix}
    \dot{h}_0(t) + \dot{h}_1(t)\xi_1 \\
    \dot{\alpha}_0(t) + \dot{\alpha}_1(t)\xi_1 \\
    \ddot{h}_0(t) + \ddot{h}_1(t)\xi_1 \\
    \ddot{\alpha}_0(t) + \ddot{\alpha}_1(t)\xi_1
\end{pmatrix} =
\begin{pmatrix}
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
    -I_\alpha k_{h0}/d & -(k_1 U^2 - m_W x_\alpha b(\bar{k}_{\alpha 0} + \sigma_1 \xi_1)/d) & -c_1 & -c_2 \\
    m_W x_\alpha b k_{h0}/d & -(k_2 U^2 + m_T(\bar{k}_{\alpha 0} + \sigma_1 \xi_1)/d) & -c_3 & -c_4
\end{pmatrix}
\begin{pmatrix}
    h_0(t) + h_1(t)\xi_1 \\
    \alpha_0(t) + \alpha_1(t)\xi_1 \\
    \dot{h}_0(t) + \dot{h}_1(t)\xi_1 \\
    \dot{\alpha}_0(t) + \dot{\alpha}_1(t)\xi_1
\end{pmatrix}
\]

\[ \dot{Y}_s = A_s(U)Y_s \]
Pitch-Plunge Airfoil

- Uncertainty in Flutter Speed due to variation in torsional stiffness – Intrusive PCE

\[ k_{\alpha 0} = \bar{k}_{\alpha 0} + \sigma_1 \xi_1 \]

\[
\begin{align*}
\sigma_1 = 5\%, & \quad U_f = 8.85 \text{ ft/sec} \\
\sigma_1 = 10\%, & \quad U_f = 8.57 \text{ ft/sec} \\
\sigma_1 = 15\%, & \quad U_f = 8.31 \text{ ft/sec}
\end{align*}
\]
Pitch-Plunge Airfoil

- Uncertainty in LCO; variation in torsional stiffness – Intrusive PCE

\[ k_{\alpha_2} = \bar{k}_{\alpha_2} + \sigma_2 \xi_2 \]

\[ \xi_2 = 1 \text{ and } \sigma_2 = 0.1 \bar{k}_{\alpha_2} \]
Normal Form of Hopf Bifurcation

Introducing a small nondimensional parameter $\epsilon$ as a bookkeeping parameter

$$U = U_f + \epsilon^2 \sigma_U U_f \quad k_{\alpha 0} = \overline{k}_{\alpha 0} + \epsilon^2 \sigma_{\alpha} \overline{k}_{\alpha 0} \quad k_{h 0} = \overline{k}_{h 0} + \epsilon^2 \sigma_{h} \overline{k}_{h 0}$$

Third-order approximate solution

$$Y(t, \sigma_U, \sigma_\alpha, \sigma_h) = \epsilon Y_1(T_0, T_2) + \epsilon^2 Y_2(T_0, T_2) + \epsilon^3 Y_3(T_0, T_2) + \ldots \quad T_m = \epsilon^m t$$

The time derivative

$$\frac{d}{dt} = D_0 + \epsilon^2 D_2 + \ldots \quad D_m = \partial / \partial T_m$$

$$\dot{Y} = A(U) Y + Q(Y, Y) + C(Y, Y, Y)$$

Order ($\epsilon$)

$$D_0 Y_1 - A(U_f) Y_1 = 0$$

Order ($\epsilon^2$)

$$D_0 Y_2 - A(U_f) Y_2 = Q(Y_1, Y_1)$$

Order ($\epsilon^3$)

$$D_0 Y_3 - A(U_f) Y_3 = -D_2 Y_1 + \sigma_U B_1 Y_1 + \sigma_h B_2 Y_1 + \sigma_\alpha B_3 Y_1 + 2 Q(Y_1, Y_2) + C(Y_1, Y_1, Y_1)$$
Normal Form of Hopf Bifurcation

where

\[ B_1 = -2k_1U_f^2 I_1 - 2k_2U_f^2 I_2, \quad B_2 = -\frac{I_a}{d} I_1 + \frac{m_W b r_0}{d} I_2, \quad B_3 = \frac{m_W b r_0}{d} I_1 - \frac{m_T}{d} I_2 \]

\[ I_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \]

- The general solution of the first-order problem

\[ Y_1(T_0, T_2) = \eta(T_2) p e^{i\omega T_0} + \overline{\eta}(T_2) \overline{p} e^{-i\omega T_0} \]

\[ D_0 Y_2 - A(U_f) Y_2 = Q(p, p) \eta^2 e^{2i\omega T_0} + 2Q(p, \overline{p}) \eta \overline{\eta} + Q(\overline{p}, \overline{p}) \overline{\eta}^2 e^{-2i\omega T_0} \]

- The general solution of the second-order problem

\[ Y_2 = \zeta_2 \eta^2 e^{2i\omega T_0} + 2\zeta_0 \eta \overline{\eta} + \zeta_2 \overline{\eta}^2 e^{-2i\omega T_0} \]
Normal Form of Hopf Bifurcation

- The third-order problem

\[ D_0 Y_3 - A(U_f)Y_3 = - \left[ D_2 \eta p - \sigma_U B_1 \eta p - \sigma_h B_2 \eta p - \sigma_\alpha B_3 \eta p - (4Q(p, \zeta_0) + 2Q(\overline{p}, \zeta_2) - 3C(p, p, \overline{p})) \eta^2 \bar{\eta} \right] e^{i\omega T_0} + cc + NST \]

- Imposing the solvability condition yields the Normal form of Hopf bifurcation

\[ D_2 \eta = \beta \eta + \Lambda \eta^2 \bar{\eta} \]

For the specific values of the airfoil geometry

\[
\mathcal{R}(\beta) = \beta_r = 3.959\sigma_U U_f - 4.1354\sigma_h \overline{k_{h0}} + 2.3984\sigma_\alpha \overline{k_{\alpha0}}
\]

\[
\mathcal{S}(\beta) = \beta_i = 2.9557\sigma_U U_f + 1.5023\sigma_h \overline{k_{h0}} + 4.0114\sigma_\alpha \overline{k_{\alpha0}}
\]

\[
\mathcal{R}(\Lambda) = \Lambda_r = 0.0055995k_{\alpha2} - 0.00072529k_{\alpha1}^2 - 0.00083071k_{h1}k_{\alpha1} - 2.19904 \times 10^{-7}k_{h2} - 1.04596 \times 10^{-7}k_{h1}^2 - 0.0779953c_s
\]

\[
\mathcal{S}(\Lambda) = \Lambda_i = 0.0093651k_{\alpha2} - 0.0011922k_{\alpha1}^2 - 0.0013891k_{h1}k_{\alpha1} - 7.98847 \times 10^{-8}k_{h2} - 3.79928 \times 10^{-8}k_{h1}^2 - 0.137028c_s
\]
Normal Form of Hopf Bifurcation

Using polar form

\[ \eta = \frac{1}{2} a \exp(i\theta) \]

- The amplitude and frequency of LCO

\[ \dot{a} = \beta_r a + \frac{1}{4} \Lambda_r a^3 \]
\[ \dot{\theta} = \beta_i + \frac{1}{4} \Lambda_i a^2 \]

- Fixed points

\[ a = 0, \quad a = \pm \sqrt{-\frac{4\beta_r}{\Lambda_r}} \]

- The type of instability can be determined.
Sensitivity Analysis

\[ \beta_r = 3.959 \sigma_U U_f - 4.1354 \sigma_h k_{h0} + 2.3984 \sigma_\alpha k_{\alpha0} \]

✓ Variations in \( k_{h0} \) and \( k_{\alpha0} \) have opposite effects on the flutter speed, \( k_{h0} \) increases it and \( k_{\alpha0} \) decreases it.

✓ The effect of variation in \( k_{h0} \) is twice that of \( k_{\alpha0} \).

✓ The perturbation analysis is only valid for small fluctuations of \( k_{\alpha0} \) and \( k_{h0} \) around their mean values.
**Sensitivity Analysis**

The dependence of the type of dynamic instability on the system parameters can be identified through

\[
\Lambda_r = 0.0055995 k_{\alpha 2} - 0.00072529 k_{\alpha 1}^2 - 0.00083071 k_{h1} k_{\alpha 1} - 2.19904 \times 10^{-7} k_{h2} - 1.04596 \times 10^{-7} k_{h1}^2 - 0.0779953 c_s
\]

For hardening springs

- The nonlinearities in the plunge stiffness are favorable in the sense that they may inhibit the occurrence of LCO below the flutter boundary and limit the exponentially growing oscillations to a periodic response *(supercritical Hopf bifurcation)*.

- The pitch stiffness cubic nonlinearity may lead to large-amplitude LCOs when transitioning through Hopf bifurcation and also induce LCO even before reaching the flutter speed if the disturbances to the system are sufficiently large *(subcritical Hopf bifurcation)*.

- Variations in different system parameters can lead to subcritical instability *(unfavorable instability)* even if the deterministic problem does not exhibit that behavior.
Remarks

- Simplification of the dynamics of an aeroelastic system.

- The principle of normal form is used to characterize the type of the dynamic instability and to analytically predict the amplitude and frequency of LCO in post-Hopf bifurcation.

- This form is used to effectively perform sensitivity analysis of system’s response to variations in its parameters.