Multiscale Computation for Flows in Heterogeneous Porous Media

Thomas Y. Hou
Applied Mathematics, Caltech

Collaborators: Y. Efendiev (TAMU), V. Ginting (Univ. of Wyoming),
I. Graham (Bath), C. C. Chu, W. Luo (Caltech)

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Introduction

• Subsurface flows and transport are affected by heterogeneities at multiple scales (pore scale, core scale, field scale).

• Because of wide range of scales direct numerical simulations are not affordable.

• Upscaling of flow and transport parameters is commonly used in practice.

• Multiscale computational methods have emerged as an effective alternative to perform upscaling for two-phase flows in strongly heterogeneous porous media.

• Multiscale methods have also been used in studying stochastic PDEs, uncertainty quantification and other related applications.
Darcy’s law and permeability

Darcy’s empirical law, 1856: The volumetric flux $u(x, t)$ (Darcy velocity) is proportional to the pressure gradient

$$u = -\frac{k}{\mu} \nabla p = -K \nabla p, \quad \text{div}(u) = f$$

where $k(x)$ is the measured permeability of the rock, $\mu$ is the fluid viscosity, $p(x)$ is the fluid pressure, $u(x)$ is the Darcy velocity, $f$ is a source or sink.
Requirements/Challenges

- Accuracy and Robustness
- Valid for different types of subsurface heterogeneity

Left: two-point variogram based. Middle: synthetic channelized. Right: channelized (North Sea).

- Applicable for varying flow scenarios
Sparse representation of multiscale solutions

Kolmogorov $n$-width
Given a Banach space $V$ and a fixed integer $n$, find the best vector subspace $V_n$ of dimension $n$ that approximates $V$. Find $V_n$ such that

$$\sup_{u \in V, \|u\|=1} \inf_{u_n \in V_n} \|u - u_n\|$$

is minimized (this number is called Kolmogorov $n$-width).

Multiscale Finite Element Methods
Babuska-Osborn (83,94), Hou and Wu (1997, JCP). Consider

$$\text{div}(k_\varepsilon(x) \nabla p_\varepsilon) = f,$$

where $\varepsilon$ is a small parameter. We represent the solution in terms of the multiscale bases, $\phi_i$, in such a way that $p_\varepsilon = \sum_{i=1}^{N} p_i \phi_i$, where

$$\text{div}(k_\varepsilon(x) \nabla \phi_i) = 0 \quad \text{in} \quad K.$$

In terms of the multiscale bases, $p_\varepsilon$ has a sparse representation.
Relation to other approaches

- Multiscale multiscale finite element methods (Babuska-Osborn, Hou and Wu, Allaire and Brizzi, Nolen-Papanicolou-Pironneau.. .)
- Multiscale finite volume methods (Jenny, Lee and Tchelepi)
- Multiscale finite element methods based on two-scale convergence (C. Schwab,...)
- Variational multiscale method and subgrid modeling (T. Hughes, F. Brezzi, T. Arbogast, G. Sangalli,...)
- Partition of Unity Methods (Babuska, Osborn, Fish, ...)
- Stochastic Finite Elements (Ghanem, Spanos, Karnidakis,...)
- Heterogeneous multiscale methods (E and Engquist ...)

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Convergence property of MsFEM

Consider $k_\epsilon(x) = k(x/\epsilon)$, where $k(y)$ is periodic in $y$. $H$ - computational mesh size.

THEOREM (Hou-Wu-Cai, 1999, Math Comp) Denote $p_{\epsilon}^H$ the numerical solution obtained by MsFEM, and $p_\epsilon$ the solution of the original problem. Then, we have for $H \gg \epsilon$,

$$\|p_\epsilon - p_{\epsilon}^H\|_{1,Q} \leq C(H + \sqrt{\frac{\epsilon}{H}}).$$

- This theorem shows that MsFEM converges to the correct solution as $\epsilon \to 0$
- The ratio $\epsilon/H$ reflects two intrinsic scales. We call $\epsilon/H$ the resonance error.
- The theorem shows that there is a scale resonance when $H \approx \epsilon$. Numerical experiments confirm the scale resonance.
- Further analysis shows that the resonance error is caused by the numerical boundary layer in the boundary corrector $\theta_\epsilon$ of the multiscale base function.
Oversampling technique

• In many cases, the boundary layer of $\theta_\varepsilon$ is thin. In [Hou-Wu, JCP, 1997], we propose an oversampling technique to sample in a domain with size larger than $H$ and use only interior sampled information to construct the basis functions.

• Let $\psi^k$ be the functions in the domain $S$,

$$\text{div}(k_\varepsilon(x) \nabla \psi^k) = 0 \text{ in } S, \quad \psi^k = \text{linear function on } \partial S, \quad \psi^k(s_i) = \delta_{ik}.$$  

• The resulting multiscale finite element method is non-confirming. Its convergence has been established in [Efendiev-Hou-Wu, SINUM 2000].
MsFEM for Interface Problems with High Contrast Media

We (Chu-Graham-Hou, 09) recently study the convergence property of the multiscale finite element method for elliptic problems with high contrast media.

\[-\nabla \cdot \alpha(x) \nabla u = f, \quad x \in \Omega,\]

where \( \alpha \) is piecewise constant: \( \alpha(x) = \tilde{\alpha} > 1 \) if \( x \) is inside an inclusion, and \( \alpha(x) = 1 \) outside any inclusion. Large contrast means \( \tilde{\alpha} \gg 1 \).
By designing an appropriate boundary condition for the multiscale bases, we prove that the MsFEM solution converges with an optimal rate independent of $\alpha$:

$$\|u - u^{MS}_H\|_{H^1(\Omega),\alpha} \leq C H,$$

where $H$ is the coarse grid, and $C$ is independent of $\alpha$, even as $\alpha \to \infty$.

This generalizes and improves the previous work of Z. Li, T. Lin and X. Wu [Numer. Math, 2003]. The convergence rate of their method depends on the contrast.

Note that the convergence rate for linear FEM is much worse and the error constant $C$ depends on $\alpha$ in an essential way.
MsFEM for Interface Problems – Continued

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Table 1: $L^2$ norm error.

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Table 2: $H^1$ semi-norm error.
Applications of MsFEM to Two-Phase Flow

There are two ways one can apply MsFEM to solve the two-phase flow:

1) Solve the pressure equation on the coarse-grid and solve the saturation equation on the fine-grid

\[- \text{div}(\lambda(S)k\nabla p) = 0\]

\[\frac{\partial}{\partial t} S + v \cdot \nabla f(S) = 0,\]

where \( v = -\lambda(S)k\nabla p \). Basis functions are updated only near sharp fronts.

2) The coarse-scale equation for the transport is obtained and coupled with MsFEM for pressure equation.
Channelized permeability fields

Benchmark tests: SPE 10 Comparative Project
Channelized reservoir

Comparison of upscaled quantities (Layer 43)
Channelized reservoir

Comparison of saturation profile at PVI=0.5: (left) fine-scale model, (right) standard MsFVEM
MsFVEM utilizing limited global information

- The numerical tests using strongly channelized permeability fields (such as SPE 10 Comparative) show that local basis functions cannot accurately capture the long-range information. There is a need to incorporate some global information.

- In [Efendiev-Ginting-Hou-Ewing, JCP 2006], we propose a modified MsFEM that uses limited global information. The main idea is to use the fine-grid solution $p^0$ at time zero to determine the boundary conditions for the multiscale bases.

- This approach is different from oversampling technique. Previous related work: J. Aarnes; L. Durlofsky et al.
MsFVEM utilizing global information

• If $p^0(x_i) \neq p^0(x_{i+1})$

$$g_i(x)|_{[x_i, x_{i+1}]} = \frac{p^0(x) - p^0(x_{i+1})}{p^0(x_i) - p^0(x_{i+1})}, \quad g_i(x)|_{[x_i, x_{i-1}]} = \frac{p^0(x) - p^0(x_{i-1})}{p^0(x_i) - p^0(x_{i-1})}.$$ 

If $p^0(x_i) = p^0(x_{i+1}) \neq 0$ then

$$g_i|_{[x_i, x_{i+1}]} = \psi_i(x) + \frac{1}{2p^0(x_i)}(p^0(x) - p^0(x_{i+1})), $$

where $\psi_i(x)$ is a linear function on $[x_i, x_{i+1}]$ such that $\psi_i(x_i) = 1$ and $\psi_i(x_{i+1}) = 0$.

• The modified MsFVEM is exact for linear elliptic problem.

• When global boundary changes, then reevaluation of the basis might be needed.
Channelized reservoir

Comparison of upscaled quantities
Channelized reservoir

Comparison of saturation profile at PVI=0.5: (left) fine-scale model, (right) modified MsFVEM
Channelized reservoir

Comparison of upscaled quantities (Layer 43, changing boundary conditions)
Channelized reservoir

Comparison of saturation profile at PVI=0.5: (left) fine-scale model (right) modified MsFVEM (changing boundary condition)
A Brief Analysis

- Main goal is to show that time-varying pressure is strongly influenced by the initial pressure field.
- Use the streamline-pressure coordinates:
  \[ \frac{\partial \psi}{\partial x_1} = -v_2, \quad \frac{\partial \psi}{\partial x_2} = v_1 \]
- Set \( \eta = \psi(x, t = 0) \) and \( \zeta = p(x, t = 0) \) and transform as follows:

- high flow channel

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• The transformed pressure equation:

\[
\frac{\partial}{\partial \eta} \left( |k|^2 \lambda(S) \frac{\partial p}{\partial \eta} \right) + \frac{\partial}{\partial \zeta} \left( \lambda(S) \frac{\partial p}{\partial \zeta} \right) = 0
\]

• The transformed saturation equation:

\[
\frac{\partial S}{\partial t} + (\mathbf{v} \cdot \nabla \eta) \frac{\partial f(S)}{\partial \eta} + (\mathbf{v} \cdot \nabla \zeta) \frac{\partial f(S)}{\partial \zeta} = 0
\]

• \(|k|^2 \lambda(S) = |k_0|^2 \lambda_0(\zeta, t)1_{Q_{1-\delta}} + |k_1|^2 \lambda_1(\eta, \zeta, t)1_{Q_\delta}, \lambda(S) = \lambda_0(\zeta, t)1_{Q_{1-\delta}} + \lambda_1(\eta, \zeta, t)1_{Q_\delta}.

• The pressure has the following expansion:

\[
p(\eta, \zeta, t) = p_0(\zeta, t) + \delta p_1(\eta, \zeta, t) + \ldots,
\]

\[
\frac{\partial}{\partial \zeta} \left( \lambda_0(\zeta, t) \frac{\partial p_0}{\partial \zeta} \right) = 0.
\]

• Modified basis functions can exactly recover the initial pressure.
Numerical confirmation

Left: Pressure and streamline function at time $t = 0.4$ in Cartesian frame. Right: pressure and streamline function at time $t = 0.4$ in initial pressure-streamline frame.
Assumption G. There exists a sufficiently smooth scalar valued function $G(\eta) (G \in C^3)$, such that

$$|p - G(p^{sp})|_{1,Q} \leq C\delta,$$

where $\delta$ is sufficiently small.

Under Assumption G and $p^{sp} \in W^{1,s}(Q) (s > 2)$, we can prove that the multiscale finite element method converges with the rate given by

$$|p - p_H|_{1,Q} \leq C\delta + CH^{1-2/s} |p^{sp}|_{W^{1,s}(Q)} \leq C\delta + CH^{1-2/s}.$$
Extensions

Assume there exists a sufficiently smooth scalar valued function \( G(\eta), \eta \in R^n \) \( (G \in C^3) \), such that

\[
|p - G(u_1, \ldots, u_n)|_{1,Q} \leq C\delta,
\]

where \( \delta \) is sufficiently small.

Let \( \omega_i \) be a patch, and define \( \phi_i^0 \) to be piecewise linear basis function in patch \( \omega_i \), such that \( \phi_i^0(x_j) = \delta_{ij} \). For simplicity of notation, denote \( u_1 = 1 \). Then, the multiscale finite element method for each patch \( \omega_i \) is constructed by

\[
\psi_{ij} = \phi_i^0 u_j
\]

where \( j = 1, \ldots, n \) and \( i \) is the index of nodes. First, we note that in each \( K \), \( \sum_{i=1}^{n} \psi_{ij} = u_j \) is the desired single-phase flow solution.

THEOREM. Assume \( u_i \in W^{1,s}(Q), s > 2, i = 1, \ldots, n \). Then

\[
|p - p_H|_{1,Q} \leq C\delta + CH^{1-2/s}.
\]
The result of H. Owhadi and L. Zhang, CPAM, 2006

\[-\text{div}(k(x)\nabla p) = f,\]

where \( f \in L^p \ (p \geq 2) \), and \( k(x) \) is rough and has some mild regularity. If \( k(x) \) is very rough, we only expect \( p \in W^{1,p} \).

Take \( u_1 \) and \( u_2 \) that satisfy

\[-\text{div}(k(x)\nabla u_i) = 0 \text{ in } Q,\]

\( u_i = x_i \) on \( \partial Q \). Then, \( p(u_1, u_2) \in W^{2,p} \) because it satisfies

\[ a_{ij} \frac{\partial^2 p}{\partial u_i \partial u_j} \approx f.\]

Owhadi and Zhang showed that the finite element method with basis functions that span \( u_1 \) and \( u_2 \) converge for all \( f \in L^p \ (p \geq 2) \).
Multiscale modeling for incompressible Navier-Stokes

Motivated by the earlier work of McLaughlin-Papanicolaou-Pironneau [SIAP, 85], we develop a systematic multiscale analysis for 3D NSE.

\[ \partial_t u^\epsilon + (u^\epsilon \cdot \nabla) u^\epsilon = -\nabla p^\epsilon + \nu \Delta u^\epsilon + f, \]
\[ \nabla \cdot u^\epsilon = 0, \quad u^\epsilon|_{t=0} = U(x) + W(x, \frac{x}{\epsilon}). \]

We look for \((u^\epsilon, p^\epsilon)\) of form \((z = \theta/\epsilon, \tau = t/\epsilon)\):

\[ u^\epsilon = \bar{u}(t, x, \tau) + w(t, \bar{\theta}, \tau, z), \quad p^\epsilon = \bar{p}(t, x, \tau) + q(t, \bar{\theta}, \tau, z). \]

Two essential new ideas:
- Reparameterize a function with infinitely many scales in frequency space and decompose it into a formally two-scale structure.
- Use a nested multiscale expansion with a multiscale phase function, so that the initial two-scale structure can be preserved dynamically.
- The interesting result is that we can derive the Smagorinsky LES model as a leading approximation of our multiscale method.
- The eddy viscosity coefficients can be computed on-the-fly by our multiscale method.
MsFEM for stochastic PDEs/inverse problems

- We have applied MsFEM to study two-phase flows with stochastic permeability distributions conditioned to production data, by using the coarse grid MsFEM computation and WCE/KLE techniques.

\[-div(\lambda(S)k\nabla p) = 0\]

\[\frac{\partial}{\partial t} S + v \cdot \nabla f(S) = 0, \quad v = -\lambda(S)k\nabla p.\]

Characterizing the heterogeneous porous media via a stochastic approach has been investigated by a number of researchers. This is important in studying and quantifying the uncertainty of the multiscale problem. Some of the pioneering contributions in this area were made by Roger Ghanem and Don Zhang.


Preconditioning of MCMC Using Coarse-Scale Models

- The problem setting: Sampling the permeability field given fractional flow measurements and measurements of the permeability field at some sparse locations.

- The inverse problem is ill-posed. This problem can be regarded as sampling the permeability field conditioning on the fractional flow data with measurement errors.

- Our goal is to sample from the conditional distribution $P(k|F)$, where $k$ is the fine-scale permeability field and $F$ is the fractional flow curve measured from the production data.

- Using the Bayes theorem we can write $P(k|F) \propto P(F|k)P(k)$, where $P(k)$ is the prior distribution of the permeability field, which is assumed to be log-normal and incorporates the information of $k$ at some sparse locations.

- We assume that the likelihood function $P(k|F) \propto \exp \left( -\frac{||F - F_k||^2}{\sigma^2_f} \right) P(k)$, where $F_k$ is computed by solving the two-phase flow equation on the fine grid.
Preconditioned MCMC – continued

Algorithm (Metropolis-Hasting MCMC [Robert-Casella, 1999])

- Step 1. At state $k_n$ generate $k$ from $q(k|k_n)$, where $q(k|k_n)$ is a general transitional probability distribution which is easy to compute and has an explicit formula.
- Step 2. Accept $k$ as a sample with probability

$$p(k_n, k) = \min \left( 1, \frac{q(k_n|k)P(k|F)}{q(k|k_n)P(k_n|F)} \right);$$

i.e., take $k_{n+1} = k$ with probability $p(k_n, k)$ and $k_{n+1} = k_n$ with probability $1 - p(k_n, k)$.

The above algorithm is very expensive because

1. The MCMC method requires thousands of iterations to converge.
2. To generate a sample, one has to solve the fine grid two-phase flow equations.
3. However, the acceptance rate of the direct MCMC method is very low, due to the large dimensionality of the permeability field. Most of the CPU time is spent on simulating the rejected samples on a fine grid.
Preconditioned MCMC – continued

- To improve the MCMC method, we would like to increase its acceptance rate by modifying the proposal distribution \( q(k|k_n) \).

- **The key idea**: The proposal distribution \( q(k|k_n) \) is adapted to the target distribution using a simplified upscaling coarse-scale model based on MsFEM.

- Using the Karhunen–Loève expansion to represent the random permeability field to reduce the dimension.

- Instead of testing each proposal by fine-scale computations directly, the algorithm first tests the proposal by the coarse-scale model.

- Only if the proposal is accepted by the coarse-scale test, then a full fine-scale computation will be conducted and the proposal will be further tested as in the direct MCMC method.

- The coarse-scale test filters the unacceptable proposals and avoids the expensive fine-scale tests for those proposals.
Acceptance rate versus different coarse-scale precisions for the preconditioned MCMC method. Single-phase flow and $\sigma_f^2 = 0.001$.

In all the simulations, we test 50,000 proposals and iterate the Markov chain 50,000 times. By using MsFEM, solving the pressure equation at each time step is about 25 times faster on the coarse grid than on the fine grid.
Fractional flow comparisons. Left: Cross-plot between the reference fractional flow and sampled fractional flows. Right: the solid line designates the fine-scale reference fractional flow, and dotted lines designate fractional flows corresponding to sampled permeability fields.
We observe that the errors of both of the Markov chains converge to a steady state within 20 accepted iterations which corresponds to 20,000 proposals.
Numerical Results
Acceptance rate versus different coarse-scale precisions of the MCMC method using $6 \times 6$ and $10 \times 10$ coarse-scale models. Anisotropic single-phase flow and $\sigma_f^2 = 0.001$. 
Numerical Results
Acceptance rate versus coarse-scale precision of the MCMC method. Two-phase flow and $\sigma_c^2 = 0.001$. 

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Numerical Results

Fractional flow errors versus accepted iterations in two phase-flow.
Conclusions

• Multiscale finite element methods provide an effective multiscale computational method for flow and transport in heterogeneous porous media.

• Resonance error can be effectively removed by the over-sampling technique.

• Long range scale interaction can be captured by incorporating limited global information into the multiscale bases.

• Flow-based adaptive coordinates offer a natural physical coordinate to capture the long range interaction in a channelized media and for incompressible flows.

• Developing a systematic and robust multiscale method for engineering applications without scale separation remains one of the most challenging problems. Some new ideas have already emerged. Much more work is needed.


