Total time \( T = A + B \), which ranges from \((3 + 4 = 7)\) to \((5 + 6 = 11)\).
Divide the sample space into \( A = 3 \), \( A = 4 \), and \( A = 5 \) (m.e. & c.e. events)

\[
\Rightarrow P(T = 7) = \sum_{n=3,4,5} P(T = 7 \mid A = n)P(A = n)
\]
\[
= \sum_{n=3,4,5} P(B = 7 - n)P(A = n)
\]
\[
= P(B = 4)P(A = 3) = 0.2 \times 0.3 = 0.06
\]

Similarly
\[
P(T = 8) = P(B = 5)P(A = 3) + P(B = 4)P(A = 4)
\]
\[
= 0.6 \times 0.3 + 0.2 \times 0.5 = 0.28
\]

\[
P(T = 9) = P(B = 6)P(A = 3) + P(B = 5)P(A = 4) + P(B = 4)P(A = 5)
\]
\[
= 0.2 \times 0.3 + 0.6 \times 0.5 + 0.2 \times 0.2 = 0.4
\]

\[
P(T = 10) = P(B = 6)P(A = 4) + P(B = 5)P(A = 5)
\]
\[
= 0.2 \times 0.5 + 0.6 \times 0.2 = 0.22
\]

\[
P(T = 11) = P(B = 6)P(A = 5)
\]
\[
= 0.2 \times 0.2 = 0.04
\]

Check: \(0.06 + 0.28 + 0.4 + 0.22 + 0.04 = 1\).

The PMF is plotted as follows:
Let $X$ be the profit (in $1000) from the construction job.

(a) $P(\text{lose money}) = P(X < 0)$
   $= \text{Area under the PDF where } x \text{ is negative}$
   $= 0.02 \times 10 = 0.2$

(b) Given event is $X > 0$ (i.e. money was made), hence the conditional probability,

$$P(X > 40 \mid X > 0) = \frac{P(X > 40 \cap X > 0)}{P(X > 0)}$$

$$= \frac{P(X > 40)}{P(X > 0)}$$

Let's first calculate $P(X > 40)$: comparing similar triangles formed by the PDF and the $x$-axis (with vertical edges at $x = 10$ and at $x = 40$, respectively), we see that

$$P(X > 40) = \text{Area of smaller triangle}$$

$$= \left(\frac{70 - 40}{70 - 10}\right)^2 \times \text{Area of larger triangle}$$

$$= 0.5^2 \left[(70 - 10) \times (0.02) + 2\right] = 0.015$$

Hence the required probability $P(X > 40 \mid X > 0)$ is

$$= 0.015 \div 0.08 = 0.1875$$
3.14 

Design life = 50 years 
Mean rate of high intensity earthquake = 1/100 = 0.01/yr 

\[ P(\text{no damage within 50 years}) = 0.99 \]

Let \( P = \) probability of damage under a single earthquake 

(a) Using a Bernoulli Sequence Model for occurrence of high intensity earthquakes, 
\[ P(\text{quake each year}) = \frac{1}{100} = 0.01 \]
\[ P(\text{damage each year}) = 0.01p \]
\[ P(\text{no damage in 50 years}) = (1 - 0.01p)^{50} \equiv 0.99 \]
Hence, \( 1 - 0.01p = (0.99)^{\frac{1}{50}} = (0.99)^{0.02} = 0.9998 \)
\[ \text{or } \quad p = 0.02 \]

(b) Assuming a Poisson process for the quake occurrence, 
Mean rate of damaging earthquake = 0.01p = 0.01x0.02 = 0.0002/yr 
\[ P(\text{damage in 20 years}) = 1 - P(\text{no damage in 20 years}) \]
\[ = 1 - \mathbf{e}^{-0.0002\times20} \]
\[ = 1 - \mathbf{e}^{-0.004} \]
\[ = 0.004 \]
3.15

Failure rate = \( \nu = 1/5000 = 0.0002 \) per hour

(a) \( P(\text{no failure between inspection}) = P(N=0 \text{ in } 2500 \text{ hr}) \)
\[
= \frac{(0.0002 \times 2500)^0 e^{-0.0002 \times 2500}}{0!} = e^{-0.5} = 0.607
\]
Hence \( P(\text{failure between inspection}) = 1 - 0.607 = 0.393 \)

(b) \( P(\text{at most two failed aircrafts among } 10) \)
\[
= P(M=0)-P(M=1)-P(M=2)
\]
\[
= \frac{10!}{0! \times 10!} (0.393)^0 (0.607)^{10} - \frac{10!}{1! \times 9!} (0.393)(0.607)^9 - \frac{10!}{2! \times 8!} (0.393)^2(0.607)^8
\]
\[
= 0.0068 - 0.044 - 0.1281 = 0.179
\]

(c) Let \( t \) be the revised inspection/maintenance interval
\( P(\text{failure between inspection}) \)
\[
= 1 - P(\text{no failure between inspection})
\]
\[
= 1 - e^{-0.0002t} = 0.05
\]
\[
-0.0002t = \ln 0.95
\]
and \( t = 256.5 \rightarrow 257 \) hours
3.22

The return period $\tau$ (in years) is defined by $\tau = \frac{1}{p}$ where $p$ is the probability of flooding per year.

Therefore, the design periods of $A$ and $B$ being $\tau_A = 5$ and $\tau_B = 10$ years mean that the respective yearly flood probabilities are

\[
P(A) = \frac{1}{5} \text{ (probability per year)} \quad P(B) = \frac{1}{10} \text{ (probability per year)}
\]

(a) $P($town encounters any flooding in a given year$)$

\[
P(A \cup B) = P(A) + P(B) - P(AB)
\]

\[
= P(A) + P(B) - P(A)P(B) \quad (\because \text{A and B s.i.})
\]

\[
= \frac{1}{\tau_A} + \frac{1}{\tau_B} - \frac{1}{\tau_A \tau_B}
\]

\[
= 0.2 + 0.1 - 0.2 \cdot 0.1
\]

\[
= 0.28
\]

(b) Let $X$ be the total number of flooded years among the next half decade. $X$ is a binomial random variable with parameters $n = 5$ and $p = 0.28$.

Hence

\[
P(X \geq 2) = 1 - P(X < 2)
\]

\[
= 1 - \binom{5}{0}(1 - 0.28)^5 - \binom{5}{1}(0.28)(1 - 0.28)^4
\]

\[
\approx 0.43
\]

(c) Let $\tau_A$ and $\tau_B$ be the improved return periods for levees $A$ and $B$, respectively. Using (i) from part (a), we construct the following table:

<table>
<thead>
<tr>
<th>New $\tau_A$ (and cost)</th>
<th>New $\tau_B$ (and cost)</th>
<th>Yearly flooding probability $\frac{1}{\tau_A} + \frac{1}{\tau_B} - \frac{1}{\tau_A \tau_B}$</th>
<th>Total cost (in million dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 ($5M)</td>
<td>20 ($10M)</td>
<td>0.145</td>
<td>5 + 10 = 15</td>
</tr>
<tr>
<td>10 ($5M)</td>
<td>30 ($20M)</td>
<td>0.130</td>
<td>5 + 20 = 25</td>
</tr>
<tr>
<td>20 ($20M)</td>
<td>20 ($10M)</td>
<td>0.0975</td>
<td>20 + 10 = 30</td>
</tr>
<tr>
<td>20 ($20M)</td>
<td>30 ($20M)</td>
<td>0.0817</td>
<td>20 + 20 = 40</td>
</tr>
</tbody>
</table>

Since the goal is to reduce the yearly flooding probability to at most 0.15, all these options will work but the top one is least expensive. Hence the optimal course of action is to **change the return periods of $A$ and $B$ to 10 and 20 years, respectively**.
3.3

(a) Applying the normalization condition \( \int_{-\infty}^{\infty} f_X(x)dx = 1 \)

\[ \Rightarrow \int_{0}^{6} c(x - \frac{x^2}{6})dx = 1 \Rightarrow c \left[ \frac{x^2}{2} - \frac{x^3}{18} \right]_{0}^{6} = 1 \]

\[ \Rightarrow c = \frac{18}{9 \times 36 - 6^3} = \frac{1}{6} \]

(b) To avoid repeating integration, let’s work with the CDF of \( X \), which is

\[ F_X(x) = \frac{1}{6} \left[ \frac{x^2}{2} - \frac{x^3}{18} \right] \]

\[ = \frac{(9x^2 - x^3)}{108} \quad \text{(for } x \text{ between 0 and 6 only)} \]

Since overflow already occurred, the given event is \( X > 4 \) (cms), hence the conditional probability

\[ P(X < 5 \mid X > 4) = \frac{P(X < 5 \text{ and } X > 4)}{P(X > 4)} \]

\[ = \frac{P(4 < X < 5)}{1 - P(X \leq 4)} \]

\[ = \frac{[F_X(5) - F_X(4)]}{[1 - F_X(4)]} \]

\[ = \frac{[(9 \times 5^2 - 5^3) - (9 \times 4^2 - 4^3)]}{[108 - (9 \times 4^2 - 4^3)]} \]

\[ = \frac{(100 - 80)}{108 - 80} = \frac{20}{28} = \frac{5}{7} \approx 0.714 \]

(c) Let \( C \) denote “completion of pipe replacement by the next storm”, where \( P(C) = 0.6 \). If \( C \) indeed occurs, overflow means \( X > 5 \), whereas if \( C \) did not occur then overflow would correspond to \( X > 4 \). Hence the total probability of overflow is (with ‘ denoting compliment)

\[ P(\text{overflow}) = P(\text{overflow} \mid C)P(C) + P(\text{overflow} \mid C')P(C') \]

\[ = P(X > 5) \times 0.6 + P(X > 4) \times (1 - 0.6) \]

\[ = [1 - F_X(5)] \times 0.6 + [1 - F_X(4)] \times 0.4 \]

\[ = (1 - 100/108) \times 0.6 + (1 - 80/108) \times 0.4 \approx 0.148 \]
(a) The mean and median of X are 13.3 lb/ft² and 11.9 lb/ft², respectively (as done in Problem 3-3-3).

(b) The event “roof failure in a given year” means that the annual maximum snow load exceeds the design value, i.e. X > 30, whose probability is

\[
P(X > 30) = 1 - P(X \leq 30) = 1 - F_X(30) = 1 - [1 - (10/30)^4]
= (1/3)^4 = 1/81 \approx 0.0123 = p
\]

Now for the first failure to occur in the 5th year, there must be four years of non-failure followed by one failure, and the probability of such an event is

\[
(1 - p)^4p = [1 - (3/4)^4]^4 \times (1/3)^4 \approx 0.0117
\]

(b) Among the next 10 years, let Y count the number of years in which failure occurs. Y follows a binomial distribution with n = 10 and p = 1/81, hence the desired probability is

\[
P(Y < 2) = P(Y = 0) + P(Y = 1)
= (1 - p)^n + n(1 - p)^{n-1}p
= (80/81)^10 + 10 \times (80/81)^9 \times (1/81)
\approx 0.994
3.20

(a) Let X be the number of accidents along the 20 miles on a given blizzard day. X has a Poisson distribution with \( \lambda_X = \frac{1}{50 \text{ miles}} \times 20 \text{ miles} = 0.4 \), hence

\[
P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-0.4} = 1 - 0.670320046 \equiv 0.33
\]

(b) Let Y be the number of accident-free days among five blizzard days. With \( n = 5 \), and \( p = \text{daily accident-free probability} = P(X = 0) \equiv 0.670 \), we obtain

\[
P(Y = 2) = \binom{5}{2} p^2 (1 - p)^3 \equiv 0.16
\]
(a) Let $X$ be the number of accidents in two months. $X$ has a Poisson distribution with

$$\lambda_X = \frac{3}{12 \text{ months}} \times 2 \text{ months} = 0.5,$$

hence

$$P(X = 1) = e^{-0.5} \times 0.5 \approx 0.303,$$

whereas

$$P(\text{2 accidents in 4 months}) = e^{-\left(\frac{3}{12}\right)(4)} \left[\left(\frac{3}{12}\right)(4)\right]^2 / 2!$$

$$= e^{-1} / 2! \approx 0.184.$$ 

No, $P(1 \text{ accidents in 2 months})$ and $P(2 \text{ accidents in 4 months})$ are not the same.

(b) 20% of all accidents are fatal, so the mean rate of fatal accidents is

$$\nu_F = \nu_x \times 0.2 = 0.05 \text{ per month}.$$ 

Hence the number of fatalities in two months, $F$ has a Poisson distribution with mean

$$\lambda_F = (0.05 \text{ per month})(2 \text{ months}) = 0.1,$$

hence

$$P(\text{fatalities in two months}) = 1 - P(F = 0) = 1 - e^{-0.1} \approx 0.095.$$
(a) Let $X$ be the total number of excavations along the pipeline over the next year; $X$ has a Poisson distribution with mean $\lambda = (1/50 \text{ miles})(100 \text{ miles}) = 2$, hence

\[
P(\text{at least two excavations})
= 1 - P(X = 0) - P(X = 1)
= 1 - e^{\lambda} (1 + \lambda) = 1 - e^{-2}(3)
\approx 0.594
\]

(b) For each excavation that takes place, the pipeline has 0.4 probability of getting damaged, and hence $(1 - 0.4) = 0.6$ probability of having no damage. Hence

\[
P(\text{any damage to pipeline} \mid X = 2)
= 1 - P(\text{no damage} \mid X = 2)
= 1 - 0.6^2 = 1 - 0.36
= 0.64
\]

Alternative method: Let $D_i$ denote “damage to pipeline in i-th excavation”; the desired probability is

\[
P(D_1 \cup D_2) = P(D_1) + P(D_2) - P(D_1 D_2)
= P(D_1) + P(D_2) - P(D_1 \mid D_2) P(D_2)
= P(D_1) + P(D_2) - P(D_1) P(D_2)
= 0.4 + 0.4 - 0.4^2 = 0.8 - 0.16
= 0.64
\]

(c) Any number ($x$) of excavations could take place, but there must be no damage no matter what $x$ value, hence we have the total probability

\[
\sum_{x=0}^{\infty} P(\text{no damage} \mid x \text{ excavations}) P(x \text{ excavations})
= \sum_{x=0}^{\infty} 0.6^x \frac{e^{-\lambda} \lambda^x}{x!}
= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(0.6 \lambda)^x}{x!}
= e^{-0.4\lambda} = e^{-0.4(2)} = e^{-0.8}
\approx 0.449
\]

Alternative method: recall the meaning of $\nu$ in a Poisson process—it is the mean rate, i.e. the true proportion of occurrence over a large period of observation. Experimentally, it would be determined by

\[
\nu = \frac{n_E}{N}
\]

where $n_E$ is the number of excavations observed over a very large number ($N$) of miles of pipeline. Since 40% of all excavations are damaging ones, damaging excavations must also occur as a Poisson process, but with the “diluted” mean rate of

\[
\nu_D = \frac{0.4n_E}{N} = 0.4\nu, \text{ hence}
\]

\[
\nu_D = (0.4)(1/50) = (1/125) \text{ (damaging excavations per mile)}
\]
Hence

\[
P(\text{no damaging excavation over a 100 mile pipeline}) = e^{-(1/125 \text{ mi.})(100 \text{ mi.})} = e^{-100/125} = e^{-0.8}
\]

\[\cong 0.449\]